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The graded algebra and the derivative $\overline{\mathcal{L}}$ of spinor fields related to the twistor equation

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Abstract

This paper presents an interesting and simple relation between twistor spinors and conformal vector fields by a finite-dimensional $\mathbb{Z}_2$-graded anticommutative algebra. Examples are given, where this algebra even is a graded Lie algebra (or a super Lie algebra in other terminology).

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0. Introduction

Twistor spinors were first introduced in mathematical physics (see [22]). In mathematics, the twistor equation appeared as an integrability condition for the canonical almost complex structure of the twistor space of an oriented four-dimensional Riemannian manifold [1]. In [18] Lichnerowicz introduced the twistor spinors as zeros of the conformally invariant twistor operator $\mathcal{D}$ and started their systematical investigation (see [19–21]). Friedrich (see [11]) studied the zeros and conformal invariants ("first integrals") of twistor spinors and their conformal relation to Killing spinors in the case of arbitrary Riemannian manifolds.

We consider Riemannian spin manifolds (M, g) of dimension $n \geq 3$. A twistor spinor is a spinor field φ satisfying the differential equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for all vector fields X, where D denotes the Dirac operator.

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In this paper we associate to two spinor fields φ and ψ a vector field V by the equation

$$g(V,X) = \mathrm{Im}\langle\varphi, X \cdot \psi\rangle$$

for all vector fields X. It can easily be shown that the vector field is conformal if φ and ψ are twistor spinors.

All conformal vector fields together with all twistor spinors form a graded algebra A, which illustrates that there is a lot of structure between these rather different geometrical objects. Graded algebras, especially graded Lie algebras became interesting for physicists in the context of supersymmetries related to particles of different statistics [7]. Such anticommutative $\mathbb{Z}_2$-graded algebras in the literature on mathematical physics also occur as conventional superconformal algebras (SCA) [9]. Such a conventional superconformal superalgebra is defined to be a $\mathbb{Z}_2$-graded Lie algebra, however the Lie algebra requirement, that means the graded Jacobi identity, is dropped. Also this is exactly the situation we obtained here. Sometimes it is required that the even part of this graded algebra contains the Virasoro algebra as a Lie subalgebra. However, there exists no universal definition, fixing what a superconformal algebra could or should not contain, however most commonly one works within the class of $\mathbb{Z}_2$-graded anticommutative algebras.

For the sake of completeness we recall in Section 1.1 some facts concerning twistor spinors on Riemannian spin manifolds. The main subjects are contained in Sections 1.2 and 2.2. First we describe the graded algebra A, where the even part consists of all conformal vector fields, whereas the twistor spinors form the odd part. We then introduce a derivative $\overline{\mathcal{L}}$ of spinor fields in the direction of conformal vector fields, which is defined according to the twistor equation. It turns out that the introduced derivative $\overline{\mathcal{L}}$ is in fact the product of a conformal vector field and a twistor spinor in the graded algebra A. Thus, one might believe that A is not only a graded algebra but also a graded Lie algebra. In Section 3.2 we give two examples of such graded Lie algebras and show in a third example, however, that this is not in general so.

1. The graded algebra of conformal vector fields and twistor spinors

1.1. Twistor spinors

Let (M,g) be an n-dimensional Riemannian spin manifold with $n \geq 3$ and let S be the spinor bundle of (M,g) for a fixed spin structure $Q \to M$. A spinor field is a smooth section $\varphi \in \Gamma(S)$ of S. We denote by $\langle\,,\,\rangle$ the standard Hermitian inner product on S. The Levi-Civita connection ∇ on (M,g) induces a covariant derivative, the so-called spinor derivative on the spinor bundle which will also be denoted by ∇. Further, let $X \cdot \varphi = \mu(X \otimes \varphi)$ be the Clifford multiplication of the vector X by the spinor φ. For the Clifford multiplication we have

$$X \cdot Y + Y \cdot X = -2g(X,Y)\,\mathrm{id}_S.$$

With respect to the Hermitian product on S the Clifford multiplication satisfies

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle .$$

Moreover, the spinor derivative has the following properties:

$$X\langle \varphi, \psi \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle ,$$

$$\nabla_X (Y \cdot \varphi) = \nabla_X Y \cdot \varphi + Y \cdot \nabla_X \varphi .$$

The Dirac operator D is the first-order differential operator defined by

$$D = \mu \circ \nabla : \Gamma(S) \to \Gamma(T^*M \otimes S) \cong \Gamma(TM \otimes S) \to \Gamma(S).$$

Here we identify the bundles T^*M and TM via the metric g. Locally the Dirac operator can be written as

$$D\varphi = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \varphi$$

for $\varphi \in \Gamma(S)$, where $e_1, \dots, e_n$ is a local orthonormal frame. The mapping $p : TM \otimes S \to TM \otimes S$ defined by

$$p(X \otimes \varphi) = X \otimes \varphi + \frac{1}{n} \sum_{j=1}^{n} e_j \otimes e_j \cdot X \cdot \varphi$$

for a local orthonormal frame $e_1, \dots, e_n$ on (M, g), is a projection of $TM \otimes S$ onto the kernel $\ker \mu$ of the Clifford multiplication. The twistor operator $\mathcal{D}$ is defined as the composition of the spinor derivative and the projection p,

$$\mathcal{D} = p \circ \nabla : \Gamma(S) \to \Gamma(T^*M \otimes S) \cong \Gamma(TM \otimes S) \to \Gamma(\ker \mu).$$

Locally the twistor operator is given by

$$\mathcal{D}\varphi = \sum_{j=1}^{n} e_j \otimes \left(\nabla_{e_j} \varphi + \frac{1}{n} e_j \cdot D\varphi \right)$$

for $\varphi \in \Gamma(S)$. A spinor field φ is called a twistor spinor if and only if $\mathcal{D}\varphi = 0$. Equivalently, φ is a twistor spinor if and only if φ satisfies the twistor equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for all vector fields X.

The twistor operator $\mathcal{D}$ is conformally invariant in the following sense: Let $\overline{g} = \lambda g$ be a conformal change of the metric g, where λ is a positive real-valued function on M, and let $^{-} : S \to \overline{S}$ denote the natural isomorphism of the corresponding spin bundles. We then have the relation

$$\overline{\mathcal{D}\varphi} = \lambda^{1/4} \overline{\mathcal{D}}(\lambda^{1/4} \overline{\varphi}) \tag{1}$$

for $\varphi \in \Gamma(S)$, where $\overline{\mathcal{D}}$ denotes the twistor operator in $\overline{S}$.

On the space $\ker \mathcal{D}$ of all twistor spinors we have two conformal invariants

$$C_\varphi = \mathrm{Re}\langle D\varphi, \varphi\rangle$$

and

$$Q_\varphi = |\varphi|^2|D\varphi|^2 - C_\varphi^2 - \sum_{j=1}^{n}(\mathrm{Re}\langle D\varphi, e_j \cdot \varphi\rangle)^2 \geq 0,$$

where $e_1, \ldots, e_n$ is an orthonormal frame on (M, g). Further, if $\varphi \in \ker \mathcal{D}$, then

$$\nabla_X(D\varphi) = \tfrac{1}{2}nL(X) \cdot \varphi$$

for all vector fields X, where L denotes the (1,1)-tensor defined by

$$L = \frac{1}{n-2}\left(\frac{R}{2(n-1)}\mathrm{id} - \mathrm{Ric}\right).$$

Here R denotes the scalar curvature and Ric is the (1,1)-Ricci tensor. Obviously, L is symmetric with respect to the metric g, i.e.

$$g(X, L(Y)) = g(L(X), Y).$$

Moreover, any twistor spinor satisfies

$$D^2\varphi = \frac{Rn}{4(n-1)}\varphi.$$

Finally, we recall that the space of all twistor spinors is finite dimensional, namely (see [11])

$$\dim_{\mathbb{C}} \ker \mathcal{D} \leq 2^{[n/2]+1}.$$

1.2. The graded algebra A

We will concern with gradings with respect to the ring $\mathbb{Z}_2 = \{0, 1\}$.

For a conformal vector field X with $\mathcal{L}_X g = 2hg$ for some function h, we call h the divergence function of X. Also for two conformal vector fields X and Y with divergence functions h_X and h_Y the commutator $[X, Y]$ is a conformal vector field with divergence function $X(h_Y) - Y(h_X)$.

Definition 1.1. Let φ and ψ be any spinor fields. We then define $\varphi \circ \psi$ to be the vector field

$$\varphi \circ \psi = \sum_{j=1}^{n} \mathrm{Im}\langle \varphi, e_j \cdot \psi\rangle e_j,$$

where $e_1, \ldots, e_n$ is a local orthonormal frame.

From $\operatorname{Im}\langle\varphi, e_j \cdot \psi\rangle = -\operatorname{Im}\overline{\langle\varphi, e_j \cdot \psi\rangle} = -\operatorname{Im}\langle e_j \cdot \psi, \varphi\rangle = \operatorname{Im}\langle\psi, e_j \cdot \varphi\rangle$ we see

$$\varphi \circ \psi = \psi \circ \varphi.$$

Definition 1.2. Let X be a conformal vector field with $\mathcal{L}_X g = 2hg$. Further, let φ be any spinor field. We then define a spinor field $X \circ \varphi$ by

$$X \circ \varphi = \nabla_X \varphi + \tfrac{1}{4}\tau(\nabla X) \cdot \varphi, \quad \text{where } \tau(\nabla X) = \textstyle\sum_{j=1}^n \nabla_{e_j} X \cdot e_j + (n-2)h$$

for a local orthonormal frame $e_1, \ldots, e_n$.

Obviously, $X \circ (f\varphi) = X(f)\varphi + fX \circ \varphi$ for any complex-valued function f.

Remark. It is clear that the definitions of $\varphi \circ \psi$ and $X \circ \varphi$ are independent of the choice of the local orthonormal frame $e_1, \ldots, e_n$. In the following we suppose that $e_1, \ldots, e_n$ is an orthonormal frame arising from one in $T_x M$ by parallel displacement along geodesics, $x \in M$. Then $\nabla e_j(x) = 0$.

The products defined above satisfy the following derivative property in relation to the Lie derivative of vector fields:

Proposition 1.3. *Let φ and ψ be any spinor fields and suppose that X is any conformal vector field. Then*

$$\mathcal{L}_X(\varphi \circ \psi) = (X \circ \varphi) \circ \psi + \varphi \circ (X \circ \psi).$$

Proof. First of all we state

$$\begin{aligned}
\nabla_X(\varphi \circ \psi) &= \sum_{j=1}^n \operatorname{Im}\langle\nabla_X\varphi, e_j \cdot \psi\rangle e_j + \sum_{j=1}^n \operatorname{Im}\langle\varphi, e_j \cdot \nabla_X\psi\rangle e_j \\
&= (\nabla_X\varphi) \circ \psi + \varphi \circ (\nabla_X\psi).
\end{aligned}$$

Furthermore, we deduce

$$\begin{aligned}
(\tau(\nabla X) \cdot \varphi) \circ \psi = &\sum_{i,j=1}^n \operatorname{Im}\langle\varphi, e_j \cdot e_i \cdot \nabla_{e_i} X \cdot \psi\rangle e_j + (n-2)h\varphi \circ \psi \\
&+ 2\sum_{i,j=1}^n \delta_{ij} \operatorname{Im}\langle\varphi, \nabla_{e_i} X \cdot \psi\rangle e_j \\
&- 2\sum_{i,j=1}^n g(\nabla_{e_i} X, e_j) \operatorname{Im}\langle\varphi, e_i \cdot \psi\rangle e_j \\
= &-\varphi \circ (\tau(\nabla X) \cdot \psi) + 2(n-2)h\varphi \circ \psi - 2nh\varphi \circ \psi \\
&+ 2\sum_{i,j=1}^n g(\nabla_{e_j} X, e_i) \operatorname{Im}\langle\varphi, e_i \cdot \psi\rangle e_j - 2\nabla_{\varphi\circ\psi} X \\
= &-\varphi \circ (\tau(\nabla X) \cdot \psi) + 2(n-2)h\varphi \circ \psi - 2nh\varphi \circ \psi
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{i,j=1}^{n} g(\nabla_{e_i}X, e_j)\,\mathrm{Im}\langle\varphi, e_i\cdot\psi\rangle e_j - 2\nabla_{\varphi\circ\psi}X \\
&+4h\sum_{i,j=1}^{n}\delta_{ij}\mathrm{Im}\langle\varphi, e_i\cdot\psi\rangle e_j \\
=&-\varphi\circ(\tau(\nabla X)\cdot\psi) - 4h\varphi\circ\psi - 4\nabla_{\varphi\circ\psi}X + 4h\varphi\circ\psi,
\end{aligned}
$$

where h is given by $\mathcal{L}_X g = 2hg$. Hence,

$$(\tau(\nabla X)\cdot\varphi)\circ\psi + \varphi\circ(\tau(\nabla X)\cdot\psi) = -4\nabla_{\varphi\circ\psi}X.$$

Together, these equations imply that

$$
\begin{aligned}
(X\circ\varphi)\circ\psi + \varphi\circ(X\circ\psi) &= (\nabla_X\varphi)\circ\psi + \varphi\circ(\nabla_X\psi) - \nabla_{\varphi\circ\psi}X \\
&= [X, \varphi\circ\psi] = \mathcal{L}_X(\varphi\circ\psi).
\end{aligned}
\qquad \Box
$$

Let B be the graded vector space which has the complexification of the Lie algebra of all vector fields on M as even part B_0 and the vector space of all spinor fields on M as odd part B_1. By means of the definitions above and setting $X\circ Y = [X, Y]$ for vector fields X and Y and $\varphi\circ X = -X\circ\varphi$ for any vector field X and any spinor field φ, B becomes a graded algebra. Since $X\circ Y = [X, Y] = -[Y, X] = -Y\circ X$, $X\circ\varphi = -\varphi\circ X$ and $\varphi\circ\psi = \psi\circ\varphi$ for homogeneous elements $X, Y \in B_0$ and $\varphi, \psi \in B_1$, B is graded anticommutative.

Next we are going to show that all conformal vector fields and the solutions of the twistor equation define a finite-dimensional graded subalgebra A of B.

Proposition 1.4. *Let φ and ψ be any twistor spinors on (M, g). Then the vector field $\varphi\circ\psi$ is a conformal vector field. In fact, we have*

$$\mathcal{L}_{\varphi\circ\psi}g = 2hg,$$

where

$$h = \frac{1}{n}\mathrm{Im}\{\langle\varphi, D\psi\rangle - \langle D\varphi, \psi\rangle\}.$$

Proof. Using the definition of $\varphi\circ\psi$ and the twistor equation, we have

$$g(\nabla_Y(\varphi\circ\psi), X) = \frac{1}{n}\mathrm{Im}\{\langle X\cdot Y\cdot D\varphi, \psi\rangle - \langle\varphi, X\cdot Y\cdot D\psi\rangle\}$$

for any vector fields X and Y. This implies

$$(\mathcal{L}_{\varphi\circ\psi}g)(X, Y) = \frac{2}{n}\mathrm{Im}\{\langle\varphi, D\psi\rangle - \langle D\varphi, \psi\rangle\}g(X, Y). \qquad \Box$$

Lemma 1.5. *Let X be a conformal vector field with divergence function h. Then*

$$\tau(\nabla X)\cdot Y = Y\cdot\tau(\nabla X) + 4hY - 4\nabla_Y X$$

for any vector field Y.

Proof. It is

$$\begin{aligned}\tau(\nabla X)\cdot Y &= \sum_{j=1}^{n} Y\cdot\nabla_{e_j}X\cdot e_j + 2\sum_{j=1}^{n} g(\nabla_{e_j}X,Y)e_j - 2\nabla_Y X + (n-2)hY\\ &= Y\cdot\tau(\nabla X) + 4hY - 4\nabla_Y X.\end{aligned}$$

□

Lemma 1.6. *Let X and Y be conformal vector fields. Then*

$$\tau(\nabla X)\cdot\tau(\nabla Y) - \tau(\nabla Y)\cdot\tau(\nabla X) = 4\sum_{j=1}^{n}\{\nabla_{[e_j,X]}Y\cdot e_j - \nabla_{[e_j,Y]}X\cdot e_j\}.$$

Remark. The curvature $\mathcal{R}^S$ of the spinor derivative defined by

$$\mathcal{R}^S(X,Y)\varphi = \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi$$

is related to the curvature $\mathcal{R}$ of the Riemannian manifold (M, g) by

$$\mathcal{R}^S(X,Y)\varphi = \frac{1}{4}\sum_{i,j=1}^{n}\mathcal{R}(X,Y,e_i,e_j)e_i\cdot e_j\cdot\varphi.$$

Proposition 1.7. *We have*

$$[X,Y]\circ\varphi = X\circ(Y\circ\varphi) - Y\circ(X\circ\varphi)$$

for any conformal vector fields X and Y and for any spinor field φ.

Proof. Using the Bianchi identity

$$\mathcal{R}(X,Y)e_j + \mathcal{R}(Y,e_j)X + \mathcal{R}(e_j,X)Y = 0,$$

we see that

$$\begin{aligned}&\sum_{i,j=1}^{n}\mathcal{R}(X,Y,e_i,e_j)e_i\cdot e_j\\ &\quad= \sum_{j=1}^{n}\mathcal{R}(Y,e_j)X + \mathcal{R}(e_j,X)Y\}\cdot e_j\\ &\quad= \sum_{j=1}^{n}\nabla_{e_j}(\nabla_X Y - \nabla_Y X)\cdot e_j + \sum_{j=1}^{n}\nabla_Y(\nabla_{e_j}X\cdot e_j)\\ &\qquad - \sum_{j=1}^{n}\nabla_X(\nabla_{e_j}Y\cdot e_j) + \sum_{j=1}^{n}\{\nabla_{[e_j,Y]}X - \nabla_{[e_j,X]}Y\}\cdot e_j\\ &\quad= \tau(\nabla[X,Y]) - (n-2)h_{[X,Y]} + \nabla_Y(\tau(\nabla X)) - (n-2)Y(h_X)\\ &\qquad - \nabla_X(\tau(\nabla Y)) + (n-2)X(h_Y) + \sum_{j=1}^{n}\{\nabla_{[e_j,Y]}X - \nabla_{[e_j,X]}Y\}\cdot e_j,\end{aligned}$$

where the functions $h_{[X,Y]}$, h_X and h_Y are the divergence functions according to the conformal vector fields $[X, Y]$, X and Y. By the previous lemma, this becomes

$$\sum_{i,j=1}^{n} \mathcal{R}(X,Y,e_i,e_j)e_i \cdot e_j = \tau(\nabla[X,Y]) + \nabla_Y(\tau(\nabla X)) - \nabla_X(\tau(\nabla Y)) \\ -\tfrac{1}{4}\{\tau(\nabla X)\cdot\tau(\nabla Y) - \tau(\nabla Y)\cdot\tau(\nabla X)\}$$

such that the equation

$$0 = \mathcal{R}^S(X,Y)\varphi + \tfrac{1}{4}\nabla_X(\tau(\nabla Y))\cdot\varphi - \tfrac{1}{4}\nabla_Y(\tau(\nabla X))\cdot\varphi - \tfrac{1}{4}\tau(\nabla[X,Y])\cdot\varphi \\ +\tfrac{1}{16}\{\tau(\nabla X)\cdot\tau(\nabla Y) - \tau(\nabla Y)\cdot\tau(\nabla X)\}\cdot\varphi$$

is satisfied for any spinor field φ. On the other hand, it is easy to see that

$$X\circ(Y\circ\varphi) - Y\circ(X\circ\varphi) - [X,Y]\circ\varphi \\ = \mathcal{R}^S(X,Y)\varphi + \tfrac{1}{4}\nabla_X(\tau(\nabla Y))\cdot\varphi - \tfrac{1}{4}\nabla_Y(\tau(\nabla X))\cdot\varphi - \tfrac{1}{4}\tau(\nabla[X,Y])\cdot\varphi \\ +\tfrac{1}{16}\{\tau(\nabla X)\cdot\tau(\nabla Y) - \tau(\nabla Y)\cdot\tau(\nabla X)\}\cdot\varphi,$$

which concludes the proof. □

Lemma 1.8. *If φ is a twistor spinor, then we have*

$$\sum_{i,j=1}^{n} \mathcal{R}(X,Y,e_i,e_j)e_i\cdot e_j\cdot\varphi = 2(X\cdot L(Y) - Y\cdot L(X))\cdot\varphi$$

for any vector fields X and Y.

Proof. Using the twistor equation and the relation

$$\nabla_X(D\varphi) = \tfrac{1}{2}nL(X)\cdot\varphi$$

for twistor spinors, we obtain

$$\nabla_X\nabla_Y\varphi = \nabla_{\nabla_X Y}\varphi - \tfrac{1}{2}Y\cdot L(X)\cdot\varphi$$

for the twistor spinor φ and for any vector fields X and Y. Thus

$$\sum_{i,j=1}^{n} \mathcal{R}(X,Y,e_i,e_j)e_i\cdot e_j\cdot\varphi = 4\mathcal{R}^S(X,Y)\varphi = 2(X\cdot L(Y) - Y\cdot L(X))\cdot\varphi. \qquad \square$$

Corollary 1.9. *Let φ be a twistor spinor and X any conformal vector field with divergence function h. Then*

$$\nabla_Y(\tau(\nabla X))\cdot\varphi = 2\{X\cdot L(Y) - Y\cdot L(X) + Y\cdot \operatorname{grad} h\}\cdot\varphi$$

for all vector fields Y.

Proposition 1.10. *Let X be any conformal vector field. Then for any twistor spinor φ the product $X\circ\varphi$ is a twistor spinor, too.*

Proof. Applying the previous lemma, we deduce

$$\nabla_Y(X \circ \varphi) = Y \cdot \left\{ -\frac{1}{2}L(X) \cdot \varphi + \frac{1}{2}\operatorname{grad} h \cdot \varphi - \frac{1}{4n}\tau(\nabla X) \cdot D\varphi - \frac{1}{n}hD\varphi \right\}$$

for any vector field Y, where h is given by $\mathcal{L}_X g = 2hg$. Setting ψ the expression in braces on the right-hand side, i.e.

$$\psi = -\frac{1}{2}L(X) \cdot \varphi + \frac{1}{2}\operatorname{grad} h \cdot \varphi - \frac{1}{4n}\tau(\nabla X) \cdot D\varphi - \frac{1}{n}hD\varphi,$$

the last equation becomes

$$\nabla_Y(X \circ \varphi) = Y \cdot \psi.$$

Therefore, the twistor equation

$$\nabla_Y(X \circ \varphi) + \frac{1}{n}Y \cdot D(X \circ \varphi) = 0$$

is satisfied. □

Consider the Lie algebra of all conformal vector fields on M and define A_0 to be the complexification of this algebra. The dimension of the Lie algebra of all conformal vector fields is less than or equal to $\frac{1}{2}(n+1)(n+2)$ [15]. Thus, A_0 is a finite-dimensional subspace of B_0. Furthermore, let $A_1 = \ker \mathcal{D} \subset B_1$ be the complex subspace of all twistor spinors on M. A_1 is also a finite-dimensional vector space, since the dimension of $\ker \mathcal{D}$ is finite. Now let A be the graded vector space $A_0 \oplus A_1$.

Theorem 1.11. *Let (M, g) be a Riemannian spin manifold admitting twistor spinors. Then the graded vector space A defined above is a finite-dimensional graded anticommutative algebra.*

Proof. This is a consequence of Propositions 1.4, 1.7 and 1.10. □

Example. For $M = \mathbb{R}^n$ the algebra A has maximal dimension (cf. [11]).

2. The derivative $\overline{\mathcal{L}}$ of spinor fields

The purpose of this section is to introduce a derivative $\overline{\mathcal{L}}$ of spinor fields in the direction of conformal vector fields. The derivative we are going to define is related to the conformal invariance of the twistor equation.

The general question of constructing a Lie derivative for spinor fields is studied for instance by Kosmann in [16] and Bourguignon and Gauduchon in [6]. The paper of Bourguignon and Gauduchon gives a geometric construction of the so-called metric Lie derivative of spinor fields. The problem is to compare spinor fields for different metrics, since a diffeomorphism F transforms the metric tensor g by $F : g \mapsto F^*g$ and the spinor fields over (M, g) will be transformed into spinor fields over (M, F^*g). But, if the metrics are conformally equivalent there exists a canonical isomorphism between the corresponding spinor bundles (see [2]) which makes it possible to define a Lie derivative for spinor fields in the classical way of defining a Lie derivative. Also one sees that the so-called metric Lie derivative of a spinor field φ with respect to the conformal vector field X, introduced in [6], here is nothing else but the Lie derivative $\mathcal{L}_X\varphi$.

2.1. The invariance of the twistor spinors under diffeomorphisms

In order to define the derivative $\overline{\mathcal{L}}$ we show first an invariance property of twistor spinors under diffeomorphisms.

Let (M, g) be an n-dimensional Riemannian spin manifold and let F be any orientation preserving diffeomorphism of M. Then F induces an isomorphism F_* of the $SO(n)$-frame bundles P^F and P according to the metric tensors F^*g and g

$$F_* : P^F \to P \qquad (e_1, \ldots, e_n) \mapsto (F_*e_1, \ldots, F_*e_n).$$

This isomorphism maps orthonormal frames with respect to F^*g to orthonormal frames for g.

Let (Q, f) be a fixed spin structure for (M, g) and let (Q^F, f') be a spin structure for (M, F^*g) such that F_* lifts to an isomorphism $\tilde{F}_* : Q^F \to Q$, i.e. such that the diagram

$$\begin{array}{ccc} Q^F & \xrightarrow{\tilde{F}_*} & Q \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ P^F & \xrightarrow[F_*]{} & P \end{array}$$

commutes. Let $S = Q \times_\rho \Delta^n$ and $S^F = Q^F \times_\rho \Delta^n$ be the corresponding spinor bundles, i.e. the vector bundles associated to the spin structures via the spinor representation $\rho : \mathrm{Spin}(n) \to GL(\Delta^n)$. A spinor field over (M, g) is a section of the spinor bundle S, or, equivalently, a ρ-equivariant map $\varphi : Q \to \Delta^n$. Now define the transformed spinor field $(F^{-1})_*\varphi$ by

$$(F^{-1})_*\varphi = \varphi \circ \tilde{F}_*,$$

where a spinor field is regarded as an equivariant map. Then $(F^{-1})_*\varphi$ is a spinor field over (M, F^*g) with respect to the spin structure (Q^F, f').

Obviously, F is an isometry between the Riemannian manifolds (M, g) and (M, F^*g). Thus,

$$\nabla^F_{(F^{-1})_*X}(F^{-1})_*Y = (F^{-1})_*(\nabla_X Y)$$

for vector fields X and Y on (M, g), where ∇^F is the Levi-Civita connection of $(M, F^* g)$. This implies that the induced spinor derivative in S^F which we also denote by ∇^F satisfies

$$\nabla^F_{(F^{-1})_* X}(F^{-1})_* \varphi = (F^{-1})_*(\nabla_X \varphi).$$

Consequently,

$$D^F((F^{-1})_* \varphi) = (F^{-1})_*(D\varphi)$$

for the Dirac operator D^F in S^F.

Finally, we have the following result.

Proposition 2.1. *If $\varphi \in \Gamma(S)$ is a twistor spinor on (M, g) and F is an orientation preserving diffeomorphism of M, then the spinor field $(F^{-1})_* \varphi \in \Gamma(S^F)$ is a twistor spinor on $(M, F^* g)$.*

2.2. Definition of $\overline{\mathcal{L}}$ and relations to the product structure in A

Let $\varphi \in \Gamma(S)$ be any non-trivial solution of the twistor equation for g. Then by Proposition 2.1, the spinor field $(\Phi_t^{-1})_* \varphi \in \Gamma(S^{\Phi_t})$ solves the twistor equation with respect to the metric $\Phi_t^* g$. Since $\Phi_t^* g = \mathrm{e}^{2\sigma_t} g$, we conclude from (1) that $\exp[-(\sigma_t/2)]\iota_t((\Phi_t^{-1})_* \varphi)$ is a twistor spinor with respect to the metric g. In fact, setting

$$\psi_t = \mathrm{e}^{-\sigma_t/2}\iota_t((\Phi_t^{-1})_* \varphi),$$

we see that

$$\overline{\mathcal{D}}\psi_t^t = \mathrm{e}^{\sigma_t/2}\overline{\mathcal{D}}^t(\mathrm{e}^{\sigma_t/2}\overline{\psi_t}^t) = \mathrm{e}^{\sigma_t/2}\overline{\mathcal{D}}^t(\overline{\iota_t((\Phi_t^{-1})_* \varphi)}^t) = \mathrm{e}^{\sigma_t/2}\overline{\mathcal{D}}^t((\Phi_t^{-1})_* \varphi) = 0,$$

where $\mathcal{D}$ is the twistor operator with respect to g, $\overline{\mathcal{D}}^t$ is the twistor operator on $(M, \Phi_t^* g)$, and $\overline{}^t : S \to S^{\Phi_t}$ denotes the natural isomorphism of the corresponding spin bundles. Further, ι_t is defined to be the inverse of $\overline{}^t$.

Let us now define the derivative $\overline{\mathcal{L}}$ by

$$\overline{\mathcal{L}}_X \varphi = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\sigma_t/2}\iota_t((\Phi_t^{-1})_* \varphi))\Big|_{t=0}.$$

One verifies that

$$\begin{aligned}\overline{\mathcal{L}}_X \varphi &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\sigma_t/2})\Big|_{t=0}\varphi + \frac{\mathrm{d}}{\mathrm{d}t}(\iota_t((\Phi_t^{-1})_* \varphi))\Big|_{t=0} \\ &= -\tfrac{1}{2}h\varphi + \mathcal{L}_X \varphi = \mathcal{L}_X \varphi - \tfrac{1}{4}((\mathcal{L}_X g)^{\#})\varphi,\end{aligned}$$

where $(\mathcal{L}_X g)^{\#}$ denotes the endomorphism given by $\mathcal{L}_X g$ and g and satisfies

$$(\mathcal{L}_X g)^{\#} = 2h\,\mathrm{id}.$$

Thus we have:

Proposition 2.2. *Let X be a conformal vector field on M and φ any twistor spinor. Then the derivative $\overline{\mathcal{L}}$ of φ in direction of X is the product of X and φ in the algebra A, i.e.*

$$\overline{\mathcal{L}}_X\varphi = X \circ \varphi.$$

Proof. This follows from

$$\mathcal{L}_X\varphi = \nabla_X\varphi + \frac{1}{8}\sum_{j=1}^{n}(\nabla_{e_j}X \cdot e_j - e_j \cdot \nabla_{e_j}X) \cdot \varphi.$$

See [16] or [6]. In the case that X is a conformal vector field with divergence function h this is equivalent to

$$\mathcal{L}_X\varphi = \nabla_X\varphi + \frac{1}{4}\sum_{j=1}^{n}\nabla_{e_j}X \cdot e_j \cdot \varphi + \frac{n}{4}h\varphi. \qquad \square$$

Now Proposition 1.7 reads as $\overline{\mathcal{L}}_{[X,Y]}\varphi = [\overline{\mathcal{L}}_X, \overline{\mathcal{L}}_Y]\varphi$. Furthermore, Proposition 1.3 would take the form

$$\mathcal{L}_X(\varphi \circ \psi) = (\overline{\mathcal{L}}_X\varphi) \circ \psi + \varphi \circ (\overline{\mathcal{L}}_X\psi)$$

or, equivalently,

$$\overline{\mathcal{L}}_X(\varphi \circ \psi) = (\mathcal{L}_X\varphi) \circ \psi + \varphi \circ (\mathcal{L}_X\psi).$$

3. Examples and applications

3.1. Imaginary Killing spinors

A spinor field $\varphi \in \Gamma(S)$ is called an imaginary Killing spinor to the Killing number $\mathrm{i}\lambda$ for $\lambda \in \mathbb{R}\setminus\{0\}$ if the differential equation

$$\nabla_X\varphi = \mathrm{i}\lambda X \cdot \varphi$$

is satisfied for all vector fields X on M. Then φ is a twistor spinor, since $D\varphi = -\mathrm{i}\lambda n\varphi$.

Proposition 3.1. *Let φ, ψ and χ be imaginary Killing spinors, all to the same Killing number* $\mathrm{i}\lambda$. *Then the spinor field $(\varphi \circ \psi) \circ \chi$ is an imaginary Killing spinor to the Killing number* $-\mathrm{i}\lambda$.

Proof. Let $X = \varphi \circ \psi$. Then X has the divergence function $h = 2\lambda\,\mathrm{Re}\langle\varphi, \psi\rangle$ with gradient $\operatorname{grad} h = 4\lambda^2 X$. One verifies that

$$\nabla_Y X = 2\lambda\,\mathrm{Re}\langle\varphi, \psi\rangle Y = hY$$

for all Y. Thus, $\tau(\nabla X) = -4\lambda\,\mathrm{Re}\langle\varphi, \psi\rangle = -2h$, which gives

$$X \circ \chi = \mathrm{i}\lambda X \cdot \chi - \tfrac{1}{2}h\chi.$$

Finally,

$$\begin{aligned}\nabla_Y(X \circ \chi) &= \mathrm{i}\lambda \nabla_Y X \cdot \chi - \lambda^2 X \cdot Y \cdot \chi - \tfrac{1}{2} g(\operatorname{grad} h, Y)\chi - \tfrac{1}{2} h \mathrm{i}\lambda Y \cdot \chi \\ &= \mathrm{i}\lambda Y \cdot \chi + \lambda^2 Y \cdot X \cdot \chi + 2\lambda^2 g(X, Y)\chi - 2\lambda^2 g(X, Y)\chi - \tfrac{1}{2} h \mathrm{i}\lambda Y \cdot \chi \\ &= -\mathrm{i}\lambda\{\mathrm{i}\lambda X \cdot \chi - \tfrac{1}{2} h \chi\} = -\mathrm{i}\lambda(X \circ \chi)\end{aligned}$$

for any vector field Y. □

Remark. There exist odd-dimensional manifolds with non-trivial Killing spinors to the Killing number $\mathrm{i}\lambda$ and no Killing spinor to the Killing number $-\mathrm{i}\lambda$ [3].

3.2. Graded Lie algebras

Going back to the graded algebra A constructed in Section 1.2 by the twistor equation, we note that $X \circ Y$ for vector fields X and Y was defined to be the Lie bracket $[X, Y]$ of these vector fields. Understanding the composition $X \circ \varphi$ of a vector field X and a spinor field φ as the derivative $\overline{\mathcal{L}}$ of φ in direction of the vector field X described in the previous section the whole arrangement suggests that the graded algebra A with the bilinear operation $\circ$ is a graded Lie algebra. For this a graded Jacobi identity must be satisfied. The graded Jacobi identity is given by four identities according to the degrees of the homogeneous elements.

We have the following three Jacobi identities. Obviously,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for the vector fields X, Y, Z in A_0. By Proposition 1.7 the equation

$$[X,Y] \circ \varphi - X \circ (Y \circ \varphi) + Y \circ (X \circ \varphi) = 0$$

is satisfied for vector fields X, Y in A_0 and any spinor field in A_1. Further, Proposition 1.3 gives

$$[X, \varphi \circ \psi] - (X \circ \varphi) \circ \psi - (X \circ \psi) \circ \varphi = 0$$

for any vector field X in A_0 and spinor fields φ, ψ in A_1.

The fourth Jacobi identity would say that

$$(\varphi \circ \psi) \circ \chi + (\psi \circ \chi) \circ \varphi + (\chi \circ \varphi) \circ \psi = 0$$

holds for spinor fields φ, ψ and χ in A_1.

In the following we give examples to discuss this fourth Jacobi identity.

Example. If X is a Killing vector field and φ is a parallel spinor field, then $X \circ \varphi$ is parallel too. Furthermore, for any parallel spinor fields φ, ψ and χ one verifies $(\varphi \circ \psi) \circ \chi = 0$. Thus, the fourth Jacobi identity for parallel spinor fields is trivial. Consequently, the subalgebra of A consisting of all Killing vector fields and parallel spinor fields is a graded Lie algebra.

Example [3]. Let (F^4, h) be a K3-Surface with the Yau-metric, and consider the warped product

$$(M^5, g) := (F^4 \times \mathbb{R}, \mathrm{e}^{-4\lambda t} h \oplus \mathrm{d}t^2) \quad \text{for } \lambda \in \mathbb{R}\backslash\{0\}.$$

Further, let $K(M^5, g)_{\mathrm{i}\lambda}$ denote the space of all Killing spinors of (M^5, g) to the Killing number $\mathrm{i}\lambda$. Then

$$\dim K(M^5, g)_{\mathrm{i}\lambda} = 2 \quad \text{and} \quad \dim K(M^5, g)_{-\mathrm{i}\lambda} = 0.$$

According to [12, Proposition 2.2], this gives

$$\ker \mathcal{D} = K(M^5, g)_{\mathrm{i}\lambda}.$$

Thus, by Proposition 3.1 we have $(\varphi \circ \psi) \circ \chi = 0$ for all $\varphi, \psi, \chi \in \ker \mathcal{D}$ and the fourth Jacobi identity is satisfied. Finally, A is a graded Lie algebra.

Counterexample. Here we will give a counterexample to the above idea, showing that the fourth Jacobi identity is not fullfilled in general.

Let $\varphi \in \Gamma(S)$ be an imaginary Killing spinor, i.e. φ satisfies the differential equation

$$\nabla_X \varphi = \mathrm{i}\lambda X \cdot \varphi$$

for a real non-zero number λ. The conformal vector field $X = \varphi \circ \varphi$ has the divergence function $h = 2\lambda|\varphi|^2$. We deduce

$$\begin{aligned}\langle X \circ \varphi, \varphi\rangle + \langle \varphi, X \circ \varphi\rangle &= \langle \nabla_X \varphi, \varphi\rangle + \langle \varphi, \nabla_X \varphi\rangle + \tfrac{1}{2}(n-2)h|\varphi|^2 \\ &\quad + \frac{1}{4}\sum_{j=1}^{n}\langle \nabla_{e_j} X \cdot e_j \cdot \varphi, \varphi\rangle + \frac{1}{4}\sum_{j=1}^{n}\langle \varphi, \nabla_{e_j} X \cdot e_j \cdot \varphi\rangle \\ &= X(|\varphi|^2) - h|\varphi|^2 = 2\lambda\{\|X\|^2 - |\varphi|^4\}.\end{aligned}$$

On the other hand, setting $V_\varphi(x) = \{X \cdot \varphi(x) : X \in T_x M\}$ for $x \in M$ with $\varphi(x) \neq 0$, one has

$$\mathrm{dist}^2(\mathrm{i}\varphi, V_\varphi) = |\varphi|^2 - \frac{\|X\|^2}{|\varphi|^2},$$

i.e.

$$\|X\|^2 - |\varphi|^4 = -|\varphi|^2 \mathrm{dist}^2(\mathrm{i}\varphi, V_\varphi).$$

Because of

$$Q_\varphi = n^2\lambda^2|\varphi|^2 \mathrm{dist}^2(\mathrm{i}\varphi, V_\varphi)$$

for the constant Q_φ (see [5, p.157]) we arrive at

$$\langle X \circ \varphi, \varphi\rangle + \langle \varphi, X \circ \varphi\rangle = -\frac{2}{\lambda n^2} Q_\varphi.$$

In [4] Baum shows that the hyperbolic space is the only complete manifold admitting imaginary Killing spinors with $Q_\varphi > 0$. Furthermore, there exist non-complete manifolds of non-constant sectional curvature carrying imaginary Killing spinors of this type [5]. Hence, for such a Killing spinor φ the expression $\langle X \circ \varphi, \varphi \rangle + \langle \varphi, X \circ \varphi \rangle$ does not vanish. It follows that

$$(\varphi \circ \varphi) \circ \varphi = X \circ \varphi \neq 0,$$

which contradicts the fourth Jacobi identity, since the fourth Jacobi identity for a spinor field φ in A_1 gives $(\varphi \circ \varphi) \circ \varphi = 0$.

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